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Self-Diffusion in One-Dimensional Lattice Gases in the Presence of an External Field[1]

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We study the motion of a tagged particle in a one-dimensional lattice gas with nearest-neighbor asymmetric jumps, with p (respectively, q), $p > q$, the probability to jump to the right (left). It was shown in Ref. 6 that the fluctuations in the position of the tagged particle behave normally; $\langle (\Delta X)^2 \rangle \sim Dt$. Here we compute explicitly the diffusion coefficient. We find $D = (1-\rho)(p-q)$, where ρ is the gas density. The result confirms some recent conjectures based on theoretical arguments and computer experiments.

KEY WORDS: Tagged particle; self-diffusion coefficient; asymmetric simple exclusion process.

1. INTRODUCTION

In this paper we study the diffusion coefficient of a tagged particle in an asymmetric nearest-neighbors lattice gas (simple exclusion process) in one dimension. In the lattice Z, particles are distributed according to a product Bernoulli measure μ_ρ with density $\rho > 0$. Each particle waits a (random) exponentially distributed time with mean 1 and then attempts to jump to its right (left) nearest neighbor with probability $p(q = 1 - p)$. When $p \neq q$ we say that an external field is present. The jump actually occurs iff the chosen site is empty.

At time 0 we put a tagged particle at the origin, i.e., we consider as initial distribution the measure $\hat{\mu}_\rho$, the Bernoulli measure μ_ρ conditioned on

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there being a particle at the origin. Let $X(t)$ be the position of the tagged particle at time t; by definition $X(0)=0$ ($\hat{\mu}_\rho$ is the equilibrium measure for the process as seen from the tagged particle[5]). The problem we study is the limiting behavior of $X(t)$ when the medium is in equilibrium. First of all one notes that the increments $X(t)-X(0)$ are stationary and a drift is present: $EX(t)=\langle X(t)\rangle=(p-q)(1-\rho)\,t$. In the usual scaling for the central limit theorem one looks at $t^{-1/2}[X(t)-(p-q)(1-\rho)\,t]$ and defines the "diffusion coefficient" D as

$$D =: \lim_{t\to\infty} \frac{1}{t} E(X(t)-E(X(t)))^2 \tag{1.1}$$

The problem is then to prove that D exists, is finite, and not zero.

In Section 2 we prove the following theorem.

Theorem 1. Let $X(t)$ $[X(0)\equiv 0]$ be the position of a tagged particle in the one-dimensional asymmetric simple exclusion process with probability $p(q=1-p)$ to jump to the right (left); $p>q$. Let $\hat{\mu}_\rho$ be the product measure with parameter ρ conditioned on there being a particle at the origin. Then

$$E(X(t))=t(1-\rho)(p-q)$$

$$D=\lim_{t\to\infty}\frac{1}{t}E(X(t)-E(X(t)))^2=(1-\rho)(p-q) \tag{1.2}$$

where E is the expectation with respect to the process with initial measure $\hat{\mu}_\rho$.

Equation (1.2) is true also if $p=q$. In this case $D=0$. This is a well-known fact. Arratia[2] showed that the right normalization for $X(t)$ in the symmetric case is $t^{1/4}$. In fact he proved a central limit theorem for $t^{-1/4}X(t)$ and he computed the exact value of $\lim_{t\to\infty} t^{-1/2}\,EX(t)^2$. In his paper (see Introduction in Ref. 2) he conjectured that in the asymmetric case D is given by $(1-\rho)(p-q)$. Furthermore computer simulations[10] (see Ref. 8 for the case $p=q$) and theoretical considerations[1,3] gave support to the conjecture, and indeed a partial proof of Eq. (1.2) has been obtained by Kutner and van Beijeren.[10] Finally, Kipnis in a recent paper[6] shows that in the limit as $t\to\infty (1/\sqrt{t})(X(t)-E(X(t)))$ goes to a Gaussian random variable with covariance $D>0$.

The proof of Theorem 1 is based on a generalization (see Theorem 2 below) of a formula for D proved in Ref. 4 in the setup of reversible Markov processes, i.e., where there is detailed balance in the stationary state. The derivation of this formula is quite elementary; we obtain it following an argument given in Ref. 3.

To explain the idea we have in mind, let us first consider the reversible case, that is, the same model described so far but with $p=q=1/2$, so $E(X(t))=0$ and $\hat{\mu}_\rho$ is reversible. The starting point is the analogy with "deterministic systems." We call here "deterministic" the systems for which the variable $X(t)$ can be written as $X(t)=\int_0^t ds\, v(s)$, where (the "velocity") $v(s)$ is a centered random variable in a reversible process. Then, defining $f(t)=E(X(t)^2)$ we use integration by part formula to write

$$f(t)=tf'(0)+\int_0^t ds\,(t-s)f''(s) \tag{1.3}$$

where f' and f'' are the first and second derivatives, respectively. Then

$$\begin{aligned} f'(t) &= \lim_{h\to 0}\frac{1}{h}E(X(t+h)^2-X(t)^2) \\ &= \lim_{h\to 0}\frac{1}{h}E(X(t+h)-X(t))^2+2\lim_{h\to 0}\frac{1}{h}E(X(t)[X(t+h)-X(t)]) \end{aligned} \tag{1.4}$$

The first limit in Eq. (1.4) is zero because $[X(t+h)-X(t)]^2=0(h^2)$ and so

$$f'(0)=0 \tag{1.5a}$$

and

$$f'(t)=2E(X(t)\,v(t)), \qquad t\neq 0 \tag{1.5b}$$

By using stationarity we have

$$\begin{aligned} f''(t) &= 2\frac{d}{dt}E([X(t)-X(0)]\,v(t)) \\ &= 2\frac{d}{dt}E([X(0)-X(-t)]\,v(0)) \\ &= 2E(v(-t)\,v(0))=2E(v(t)\,v(0)) \end{aligned} \tag{1.6}$$

Combining Eqs. (1.3), (1.5), and (1.6) we obtain

$$E(X(t)^2)=2\int_0^t ds\,(t-s)\,E(v(s)\,v(0)) \tag{1.7}$$

We now divide by t and take the limit $t\to\infty$. Since the process is reversible $E(v(s)\,v(0))\geqslant 0$, hence, by the monotone convergence theorem, we get

$$D=2\int_0^\infty ds\,E(v(s)\,v(0)) \tag{1.8}$$

Now one has two problems, to show both that

(A) $D < \infty$, that is $\int_0^\infty ds\, E(v(s)\, v(0)) < \infty$; and
(B) $D > 0$.

When the system is not deterministic, the situation is quite different. Let us come back to our symmetric lattice gas model. Here we quote Ref. 3: "A small conceptual problem is that in this model under consideration the particles are assumed to jump instantaneously, hence the velocity $\cdots$ is not defined in the usual sense." We then define an effective velocity v_{eff} as

$$\lim_{h \to \infty} \frac{1}{h} E(X(h) - X(0))/\mathscr{F}_0) = v_{\text{eff}}(0) \tag{1.9}$$

where $\mathscr{F}_0$ is the σ-algebra that gives the specification of the past until time 0. Obviously, in the deterministic case $v_{\text{eff}} = v$. In the symmetric simple exclusion, $v_{\text{eff}}(0)$ is given by the rate of jumps to the right minus that to the left, hence

$$v_{\text{eff}}(0) = \tfrac{1}{2}\{[(1 - \eta_0(1)] - [1 - \eta_0(-1)]\} = \tfrac{1}{2}[\eta_0(-1) - \eta_0(1)] \tag{1.10}$$

where $\eta_t(x) = 1$ (0) if an time t there is (is not) a particle at site $x \in Z$.

We can now proceed as before. Starting from the identity (1.3) we notice that Eq. (1.4) is still true. However, the first limit in the right-hand side of Eq. (1.4) does not vanish; indeed it is easy to see that

$$\lim_{h \to 0} \frac{1}{h} E(X(t+h) - X(t))^2 = \tfrac{1}{2}E([1 - \eta_t(1)] + [1 - \eta_t(-1)]) = 1 - \rho > 0 \tag{1.11}$$

Here stationarity has been used. The limit in Eq. (1.11) is usually referred to as a martingale contribution. As we learn from Ito calculus we cannot generally neglect the square increment of our variable since, like in Brownian motion, the increments are of order $\sqrt{t}$ and not t. On the other hand the second limit in the right-hand side of Eq. (1.4) gives the same answer since Eq. (1.9) holds. So in this case [compare to Eq. (1.5)]

$$f'(t) = (1 - \rho) + 2E(X(t)\, v_{\text{eff}}(t)) \tag{1.12}$$

By taking the derivative of Eq. (1.12) we do not get the same answer as in Eq. (1.6), in fact there is still another very crucial difference: as in Eq. (1.6) we have to compute

$$f''(t) = 2\frac{d}{dt} E([X(0) - X(-t)]\, v_{\text{eff}}(0)) \tag{1.13}$$

but Eq. (1.9) does not help us anymore: we need to introduce also the backward effective velocity, that is

$$\lim_{h\to 0} \frac{1}{h} E(X(-h) - X(0)/\tilde{\mathscr{F}}_0) = \tilde{v}_{\text{eff}}(0) \tag{1.14}$$

where $\tilde{\mathscr{F}}_0$ now is the σ-algebra that gives the specification of the future starting from time zero.

In the symmetric simple exclusion, it is easy to see that

$$\tilde{v}_{\text{eff}}(t) = v_{\text{eff}}(t) \qquad \forall t \tag{1.15}$$

We observe that in the deterministic case $\tilde{v}_{\text{eff}} = -v$.

From Eqs. (1.14) and (1.15) we get

$$f''(t) = 2\frac{d}{dt} E([X(0) - X(-t)]\, v_{\text{eff}}(0)) = -2E(v_{\text{eff}}(0)\, v_{\text{eff}}(t)) \tag{1.16}$$

Hence the final formula for D is

$$D = 1 - \rho - 2\int_0^{\infty} ds\, E(v_{\text{eff}}(s)\, v_{\text{eff}}(0)) \tag{1.17}$$

Comparing the formula Eq. (1.16) with that obtained in the deterministic case Eq. (1.8) we notice that we do not have to worry about problem (A) because Eq. (1.17) automatically shows the finiteness of both D and the integral of the velocity autocorrelation function. We want to emphasize that Eq. (1.17) makes life very easy: all that is left to prove is that $D > 0$! In Ref. 4 there has been derived under very general conditions [more or less the existence of the limits in Eqs. (1.9) and (1.11) and the validity of Eq. (1.15)] in the context of reversible Markov processes a formula like Eq. (1.17), and convergence to Brownian motion was also proved (invariance principle).

Now let us come back to our process. In the symmetric case Arratia[2] proved that $D = 0$, as we pointed out before. One may check it directly from Eq. (1.17). A formula analogous to Eq. (1.17) holds also in the asymmetric case. Everything is the same except that Eq. (1.15) does not hold and we are left with

$$E(X(t) - E(X(t))^2) = t(1-\rho) - 2\int_0^{t} ds\, (t-s)\, E(\tilde{v}_{\text{eff}}(0)\, v_{\text{eff}}(s)) \tag{1.18}$$

Here we do not have automatically the convergence of the integral because we do not know whether the integrand has a definite sign. We will find that

$$\int_0^{\infty} ds\, |E(\tilde{v}_{\text{eff}}(0)\, v_{\text{eff}}(s)| < \infty \tag{1.19}$$

by actually computing the integral and as a consequence we get the exact value of D in this case.

The above arguments prove the following theorem.

Theorem 2. Let $\{\xi_t, t \in R\}$ be a Markov process with equilibrium measure μ. Let $X(t)$ be a centered process on the path space measurable with respect to $\mathscr{F}_t = \sigma$-algebra generated by $\{\xi_s, s \leqslant t\}$. Assume $\mu(X(0)=0)=1$. Assume the following three limits exist finite in $L^2(\mu)$:

$$\lim_{h\to 0} \frac{1}{h} E(X(h)/\mathscr{F}_0) = \phi_+(\xi), \qquad \phi_+ \in L^2(\mu) \tag{1.20a}$$

$$\lim_{h\to 0} \frac{1}{h} E(X(h))^2 = C \tag{1.20b}$$

$$\lim_{h\to 0} \frac{1}{h} E(X(-h)/\tilde{\mathscr{F}}_0) = \phi_-(\xi), \qquad \phi_- \in L^2(\mu) \tag{1.20c}$$

where $\tilde{\mathscr{F}}_0$ is the σ-algebra of the future, that is, $\tilde{\mathscr{F}}_0$ is generated by $\{\xi_s, s > 0\}$. Assume also that

$$\int_0^\infty ds\, |E(\phi_-(\xi_0)\, \phi_+(\xi_s))| < \infty \tag{1.21}$$

Then the following limit exists and it is finite:

$$D = \lim_{t\to\infty} \frac{1}{t} E(X(t)^2)$$

Furthermore

$$D = C - 2\int_0^\infty ds\, E(\phi_-(\xi_0)\, \phi_+(\xi_s)) \tag{1.22}$$

where C is given in Eq. (1.20).

2. PROOF OF THEOREM 1

The proof of Theorem 1 is a trivial consequence of Theorem 2 and Lemma 1, Section 1 of Ref. 6 as we shall see in a while. We first need to introduce some notation and definitions. We use the standard notation of lattice gas models, so $\eta \in \{0, 1\}^Z$ denotes a configuration of particles, $\eta(x) = 1$ (respectively, 0) if the site $x \in Z$ is occupied (empty). We describe the system as seen from the tagged particle, so the state space is

$\mathscr{X} = \{\eta \in \{0, 1\}^Z : \eta(0) = 1\}$. The generator of the dynamics is given by the sum of two generators (f is a cylindrical function):

$$Lf(\eta) = L_1 f(\eta) + L_0 f(\eta) \tag{2.1a}$$

where L_0 describes the shifts of the system due to the motion of the tagged particle and L_1 describes the motion of the other particles. More precisely

$$\begin{aligned} L_1 f(\eta) = \sum_{x \neq 0} & \eta(x)[1 - \eta(x+1)]\, p[f(\eta^{x,x+1}) - f(\eta)] \\ & + \eta(x)[1 - \eta(x-1)]\, q[f(\eta^{x,x-1}) - f(\eta)] \end{aligned} \tag{2.1b}$$

$$\begin{aligned} L_0 f(\eta) = {} & [1 - \eta(1)]\, p[f(\tau_1 \eta^{0,1}) - f(\eta)] \\ & + [1 - \eta(-1)]\, q[f(\tau_{-1} \eta^{0,-1}) - f(\eta)] \end{aligned} \tag{2.1c}$$

where $\eta^{x,y}(z)$ denotes the configuration η with the occupation number in x and y interchanged and τ_x is the shift by x, i.e., $(\tau_x \eta)(z) = \eta(z + x)$.

L generates a Feller process on $\mathscr{X}$. We will consider the process at the equilibrium with starting measure the Bernoulli measure μ_ρ with density $\rho \in (0, 1)$ and conditioned to have a particle at the origin; cf. Ref. 5 for instance. Obviously, the position $X(t)$ of the tagged particle is given by the number of shifts of the process η_s up to time t. We label the particles of the initial configuration η according to their order. So we get a sequence $\{x_i\}_{i \in Z}, x_0 = 0$ and x_i is the position of the ith particle. Because of the nearest-neighbor assumption the particles keep their labels and $x_i(t) (i \in Z)$ denotes the position at time t of the ith particle, $x_i(0) = x_i$. The process of the number of consecutive empty sites to the right of each particle is a zero range process.[7,9] The state space is N^Z, the configuration $\xi_t \in N^Z$ is defined by $\xi_t(u) =$ number of successive empty sites to the right of $x_u(t)$. The generator L' of the process ξ_t is given by (g is a bounded cylindrical function on N^Z)

$$L'g(\xi) = \sum_{u \in Z} 1\{\xi(u) \geqslant 1\}(p[g(\xi^{u,u-1}) - g(\xi)] + q[g(\xi^{u,u+1}) - g(\xi)]) \tag{2.2a}$$

where $1\{\cdot\}$ is the characteristic function of the set $\{\cdot\}$ and

$$\xi^{u,v}(z) = \begin{cases} \xi(z), & z \neq u, v \\ \xi(u) - 1, & z = u \\ \xi(v) + 1, & z = v \end{cases} \tag{2.2b}$$

That is, with rate one from each nonempty site a particle jumps to the right (left) with probability q (p). [Note: q (p) to the right (left).] The measure ν_ρ

corresponding to $\hat{\mu}_\rho$ is a product measure with geometric distribution, namely, $\nu_\rho(\xi(x)=k)=(1-\rho)^k\rho, x\in Z$. We will use later that $\nu_\rho(\xi(x)>0)=1-\rho$. Define $N(t)$ to be the difference between (1) the number of particles which jumped in $[0,t]$ from site 0 to site-1 and (2) that of the particles which jumped from site -1 to site 0.

$N(t)$ is a process adapted to ξ_t and, by definition, $N(t)=X(t)$ the position of the tagged particle in the simple exclusion process.

Proof of Theorem 1. To prove Theorem 1 we use Theorem 2 with respect to the $N(t)$ process. We have to show that the limits in Eqs. (1.19) exist. First of all we observe that

$$E(N(t))=t(p-q)(1-\rho)$$

Furthermore one easily checks that

$$\phi_+(\xi_t)=\lim_{h\to 0}\frac{1}{h}E(N(t+h)-N(t)/\mathscr{F}_t)=1\{\xi_t(0)>0\}\,p \\ -1\{\xi_t(-1)>0\}\,q \quad (2.3a)$$

$$C=E(\lim_{h\to 0}\frac{1}{h}E(N(h)^2/\mathscr{F}_0))=E(1\{\xi_t(0)>0\}\,p+1\{\xi_t(-1)>0\}\,q) \\ =(p+q)(1-\rho)=(1-\rho) \quad (2.3b)$$

$$\phi_-(\xi_t)=\lim_{h\to 0}\frac{1}{h}E(X(t-h)-X(t)/\tilde{\mathscr{F}}_t)=-1\{\xi_t(-1)>0\}\,p \\ +1\{\xi_t(0)>0\}\,q \quad (2.3c)$$

where as in Theorem 2, $\mathscr{F}_t$ is the σ-algebra generated by $\{\xi_s; s\leqslant t\}$ while $\tilde{\mathscr{F}}_t$ is that generated by $\{\xi_s; s>t\}$.

We apply Theorem 2 to the following random variable on the path space

$$N_0(t)=N(t)-t(1-\rho)(p-q) \quad (2.4)$$

So, the diffusion coefficient defined as $D=\lim_{t\to\infty}(1/t)\,E(N_0(t)^2)$ is given by

$$D=(1-\rho)-2\int_0^\infty ds\,[E(\phi_-(\xi_0)\,\phi_+(\xi_s))-(1-\rho)^2(p-q)^2] \quad (2.5)$$

provided that Eq. (2.6) below is true

$$\int_0^\infty ds\,|E(\phi_-(\xi_0)\,\phi_+(\xi_s))-(1-\rho)^2(p-q)^2|<\infty \quad (2.6)$$

We prove Eq. (2.6) by computing the exact value of the integral in the right-hand side of Eq. (2.5).

The starting point is the following identity:

$$\int_0^\infty ds\, E(\phi_-(\xi_0)\,\phi_+(\xi_s) - (1-\rho)^2(p-q)^2)$$
$$= -p^2 A(1) - q^2 A(-1) + 2pqA(0) \tag{2.7}$$

where

$$A(x) = \int_0^\infty ds\, E(1\{\xi_s(x) > 0\}\, 1\{\xi_0(0) > 0\} - (1-\rho)^2) \tag{2.8}$$

In Ref. 6 it is proven that $A(x)$ is well defined and its value explicitly computed, cf Lemma 1 of Section 1 of Ref. 6. Unfortunately this is not evident at first sight because of the different notation we are using. For the convenience of the reader we briefly report some details.

We rewrite $A(x)$ in the following way:

$$A(x) = \int_0^\infty ds\, [P(\{\xi_s(x) > 0\}/\{\xi_0(0) > 0\}) - P(\{\xi_s(x) > 0\})]\, P(\{\xi_0(0) > 0\})$$
$$= (1-\rho)\int_0^\infty ds\, [\hat{P}(\{\xi_s(x) > 0\}) - P(\{\xi_s(x) > 0\})] \tag{2.9}$$

where $\hat{P}(\{\cdot\})$ [respectively, $P(\{\cdot\})$] is the probability of the event $\{\cdot\}$ with respect to the zero range process $\hat{\xi}_t$ (respectively, ξ_t) with starting measure $\hat{\nu}_\rho(\cdot) = \nu_\rho(\cdot/\{\xi(0) > 0\})$ (respectively, ν_ρ). Since ν_ρ is a geometric measure, it is not difficult to construct a measure γ_ρ on $(N^Z)^2$ with support on the following set of configurations:

$$\bar{\mathscr{X}} = \{(\xi, \hat{\xi})\colon \xi(z) = \hat{\xi}(z), \qquad z \neq 0; \qquad \hat{\xi}(0) = \xi(0) + 1\} \tag{2.10}$$

and with marginal distributions ν_ρ and $\hat{\nu}_\rho$. We construct a coupling between the process ξ_t and $\hat{\xi}_t$ (with law P and $\hat{P}$, respectively). We require that the law $\bar{P}$ of such a coupling satisfies conditions (a), (b), and (c) below:

a. The initial distribution is γ_ρ.
b. The marginals distributions are P and $\hat{P}$
c. $\forall t \geqslant 0$ the number of sites z such that $\xi_t(z) \neq \hat{\xi}_t(z)$ is one. (That is, all the particles in the $\hat{\xi}_t$ process move as those of the ξ_t process except for the extra particle.)

The only difference between the two processes is the extra particle, so once we define its law $\bar{P}$ will be completely determined. To satisfy conditions (b) and (c) we impose that the extra particle jumps only if it is alone at its site. Given that the extra particle is the only one present at its site, it waits an exponentially distributed random time of parameter 1 and then, finally, it jumps to the right (left) with probability $q(p)$. Define the random variable $z(t)$ as the position at time t of this extra particle. So we have defined the joint law of $\xi(t)$ and $z(t)$. Now define $\hat{\xi}_t(x) = \xi_t(x) + 1\{z(t) = x\}$. In this way the coupling $\bar{P}$ is well defined and satisfies (a)–(c) above. Let $\bar{E}$ be the expectation with respect to the coupling $\bar{P}$. Let $\{S(n), n = 1, 2,...\}$ be the following sequence of stopping times $[S(0) \equiv 0]$:

$$S(n) = \inf \{t > S(n-1); |z(t) - z(S(n-1))| = 1\} \tag{2.11}$$

That is, $S(n)$ is the stopping time corresponding to the nth jump of the extra particle. It follows from the definition of the coupling $\bar{P}$ that

$$\bar{E}\left(\int_0^\infty dt\, 1\{S(n) \leqslant t < S(n+1)\}\, 1\{\xi_t(z(S(n))) = 0\}/\mathscr{F}_n\right) = 1 \tag{2.12}$$

where $\mathscr{F}_n$ is the σ-algebra generated by $\{(\xi_t, z_t), t \leqslant S(n)\}$ and $\bar{E}(\cdot/\mathscr{F}_n)$ is the conditional expectation.

Equation (2.12) plays a key role. Its proof is straightforward: the integrand in Eq. (2.12) is simply the total amount of time that the extra particle waits between two jumps when no other particles are present. By definition this random time is exponentially distributed with parameter 1, hence its expectation equals one. Now we come back to the computation of $A(x)$ [see Eq. (2.9)]:

$$\begin{aligned} A(x) &= (1-\rho)\int_0^\infty ds\, \bar{E}(1\{\hat{\xi}_s(x) > 0\} - 1\{\xi_s(x) > 0\}) \\ &= (1-\rho)\int_0^\infty ds\, \bar{E}(1\{z(s) = x\}\, 1\{\xi_s(x) = 0\}) \\ &= (1-\rho)\sum_{n=0}^\infty \bar{P}(z(S_n) = x) \\ &\quad \times \bar{E}\left(\int_0^\infty 1\{S(n) \leqslant s < S(n+1)\}\, 1\{\xi_s(x) = 0\}/\mathscr{F}_n\right) \\ &= (1-\rho)\sum_{n=0}^\infty p(n, x) \end{aligned} \tag{2.13}$$

where $p(n, x)$ is the probability of hitting x at time n for a discrete asymmetric random walk which jumps with probability $q(p)$ to the right (left). The value of the sum in the right-hand side of Eq. (2.13) is

$$\sum_{n=0}^{\infty} p(n,x) = \begin{cases} \dfrac{1}{p-q}, & x \leqslant 0 \\ \left(\dfrac{q}{p}\right)^x \dfrac{1}{p-q}, & x > 0 \end{cases} \tag{2.14}$$

cf. Lemma 1 of Section 1 of Ref. 6. Substituting Eq. (2.14) into Eqs. (2.7) and (2.5) we have

$$D = (1-\rho)(p-q)$$

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